On algebraic and geometric properties of spectral convex bodies

Raman Sanyal
Goethe-Universität Frankfurt

joint work with James Saunderson (Monash University)

Between you and I

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Focus today:

- Operations and metric properties
  Minkowski sums, polarity, volume and Steiner polynomials
- Geometric and algebraic boundary
  faces, algebraic degree, hyperbolicity
- Representations
  spectrahedra and spectrahedral shadows
Spectral convex sets

$\mathfrak{S}_d$ group of permutations acting on $\mathbb{R}^d$

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Symmetric matrices \( S_2 \mathbb{R}^d = \{ A \in \mathbb{R}^{d \times d} : A^t = A \} \).
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Spectrum \( \lambda(A) = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d \) are the \( d \) real eigenvalues of \( A \in S_2 \mathbb{R}^d \).
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A spectral convex set is a set of the form

\[ \Lambda(K) \ := \ \{ A \in S_2\mathbb{R}^d : \lambda(A) \in K \} , \]

where $K$ is a symmetric convex set.
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Proposition

If $K$ is a symmetric convex set/body, then $\Lambda(K)$ is a convex set/body.
Examples: $\Lambda(K) := \{ A \in S_2 \mathbb{R}^d : \lambda(A) \in K \}$

- Operator norm: $K = \{ x : \|x\|_\infty \leq 1 \} = [-1, 1]^d$

  $$\Lambda(K) = \{ A \in S_2 \mathbb{R}^d : \lambda_{\text{max}}(A) \leq 1 \}$$
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- **PSD cone:** $K = \mathbb{R}^d_{\geq 0}$
  
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- **Schur-Horn orbitopes** [S-Sottile-Sturmfels’11]: $K = \text{conv}(\mathcal{S}_d \cdot p)$
  \[
  \Lambda(K) = \{ A \in S_2 \mathbb{R}^d : \lambda(A) \text{ majorized by } p \} 
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$D : S_2 \mathbb{R}^d \rightarrow \mathbb{R}^d$ diagonal projection $D(A) = (A_{11}, A_{22}, \ldots, A_{dd})$

$\delta : \mathbb{R}^d \rightarrow S_2 \mathbb{R}^d$ diagonal embedding

Lemma
Let $K$ be a symmetric convex set. Then

$$D(\Lambda(K)) = K = D(\Lambda(K) \cap \delta(\mathbb{R}^d)).$$

[Needs Schur’s insight: $D(A)$ is majorized by $\lambda(A)$.]
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- $A = \sum_i \mu_i A_i$ for $A_i \in \Lambda(K)$ implies $p = D(A) = \sum_i \mu_i D(A_i) \in K$. 
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- Again by Lemma: $A = \delta(p) \in \Lambda(K)$. 

Rich class of convex sets

Let $K, L \subset \mathbb{R}^d$ be symmetric closed convex sets.

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  $$\Lambda(K) \cap \Lambda(L) = \Lambda(K \cap L).$$
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  [Hint: $K \vee L = \bigcup_{0 \leq \mu \leq 1}(1 - \mu)K + \mu L$. ]
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- **Polarity**: $K^\circ = \{ c \in \mathbb{R}^d : \langle c, x \rangle \leq 1 \text{ for all } x \in K \}$
  \[ \Lambda(K)^\circ = \Lambda(K^\circ). \]
Support functions

For closed convex set $K \subset \mathbb{R}^d$, the support function is $h_K : \mathbb{R}^d \to \mathbb{R}$

$$h_K(c) := \max\{\langle c, x \rangle : x \in K\}$$
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$h_K$ encodes $K$: $K = \{x : \langle c, x \rangle \leq h_K(c) \text{ for all } c\}$. 

Frobenius inner product on $S_2^{\mathbb{R}^d}$:

$$\langle A, B \rangle = \text{tr}(AB).$$

**Lemma**

If $K$ is closed symmetric convex set and $B \in S_2^{\mathbb{R}^d}$, then $h_{\Lambda(K)}(B) = h_K(\lambda(B))$.

$h_{\Lambda(K)}$ is a spectral convex function in the sense of [Lewis'96].

**Minkowski sum:** $\Lambda(K) + \Lambda(L) = \Lambda(K + L)$

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- Minkowski sum: $\Lambda(K) + \Lambda(L) = \Lambda(K + L)$

$$h_{K+L} = h_K + h_L$$

- Polarity: $\Lambda(K)^\circ = \Lambda(K^\circ)$

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Computation of convex invariants

$K \subset \mathbb{R}^d$ convex body, $B_d = B(\mathbb{R}^d)$ unit ball

Steiner polynomial

$$\text{vol}(K + t \cdot B_d) = \sum_{i=0}^{d} \binom{d}{i} W_{d-i}(K)t^i$$
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**Quermassintegrals** \( W_i(K) \sim \) expected volume of projection onto \((d-i)\)-flat

\( W_d \) volume, \( W_{d-1} \) surface area, \( W_1 \) mean width, \( W_0 \) Euler characteristic

Important invariants, hard to compute
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Theorem

$$\text{vol}(\Lambda(K) + t \cdot B(S_2 \mathbb{R}^d)) = 2^{\frac{d(d+3)}{2}} \prod_{r=1}^{d} \frac{\pi^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right)} \int_{K+tB_d} \prod_{i<j} |p_i - p_j| dp$$

If $K$ symmetric polytope, then integral effectively computable.
Faces and boundaries

(Exposed) Face of full-dimensional convex body $K \subset \mathbb{R}^d$

$$K^c := \{ x \in K : \langle c, x \rangle = h_K(c) \} \quad \text{for some } c \in \mathbb{R}^d$$
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$\mathcal{S}_d$-orbits of faces of $K \overset{1\text{-to}-1}{\longleftrightarrow} O(d)$-orbits of faces of $\Lambda(K)$.
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Algebraic boundary \( \partial_{\text{alg}} K \) is Zariski closure of \( \partial K \), given by \( f_K \in \mathbb{R}[x_1, \ldots, x_d] \).
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$\det(A + tl) = t^d + \eta_1(A)t^{d-1} + \cdots + \eta_d(A)$

$\eta_i(A)$ fundamental invariants of $O(d)$-action on $S_2\mathbb{R}^d$ (sums of principal minors)
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*(Exposed) Face* of full-dimensional convex body $K \subset \mathbb{R}^d$

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Corollary

$\partial_{\text{alg}} \Lambda(K)$ and $\partial_{\text{alg}} K$ have same degree.

If $K$ hyperbolicity cone, then $\Lambda(K)$ is hyperbolicity cone [Bauschke et al.’01]
Spectrahedra

Spectrahedron

\[ S = \{ x \in \mathbb{R}^d : B(x) := B_0 + x_1 B_1 + \cdots + x_d B_d \succeq 0 \} , \]

where \( B_0, \ldots, B_d \in S_2 \mathbb{R}^d \) and \( \succeq 0 \) means positive semidefinite.
Spectrahedra

Spectrahedron

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Semidefinite programming

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\begin{align*}
\max & \quad \langle c, x \rangle \\
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For example operator/nuclear/Frobenius unit balls are spectrahedra

Theorem

*If \( K \) is a polyhedron, then \( \Lambda(K) \) is a spectrahedron.*
Schur-Horn orbitopes are spectrahedra

Let \( p = (p_1 \geq p_2 \geq \cdots \geq p_d) \)

Permutahedron \( \Pi(p) = \text{conv}\{(p_{\sigma(1)}, \ldots, p_{\sigma(d)}): \sigma \in \mathfrak{S}_d\} \).
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Proposition

$x \in \Pi(p)$ if and only if $x$ majorized by $p$. That is, $\sum_i x_i = \sum_i p_i$

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\( k \)-th linearized Schur functor: linear map \( \mathcal{L}_k : S_2\mathbb{R}^d \to S_2(\wedge^k \mathbb{R}^d) \)
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Theorem (S-Sottile-Sturmfels’11)

Let \( A \in S_2 \mathbb{R}^d \). Then \( A \in \Lambda(\Pi(p)) \) if and only if \( \text{tr}(A) = \sum_i p_i \) and

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Schur-Horn orbitope is a spectrahedron of order $2^d - 2$.

For $p$ generic, algebraic degree of $\Pi(p)$ is $2^d - 2$. 
Spectral polyhedra

\( a \in \mathbb{R}^d, \ b \in \mathbb{R} \) define

\[ P_{a,b} = \{ x \in \mathbb{R}^d : \langle \sigma \cdot a, x \rangle \leq b \text{ for all } \sigma \in S_d \} . \]
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Symmetric polyhedra are of the form \(P = \bigcap_i P_{a_i,b_i}\)

Since \(\Lambda(K \cap L) = \Lambda(K) \cap \Lambda(L)\), suffices to show that \(\Lambda(P_{a,b})\) is spectrahedron.
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For \( a \in \mathbb{R}^d \), is there a linear map \( \mathcal{L}_a : S_2\mathbb{R}^d \to S_2\mathbb{R}^{d!} \) such that the \( d! \) eigenvalues of \( \mathcal{L}_a(A) \) are \( \langle \sigma \cdot a, \lambda(A) \rangle \) for \( \sigma \in \mathcal{G}_d \)?
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Yes, for \( d = 2 \): \( A \mapsto \mathcal{L}_a(A) = a_1 A + a_2 A^{\text{adj}} \)
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Assume $a = (a_1 \geq a_2 \geq \cdots \geq a_d)$

numerical chain $\mathcal{I} = (l_1, l_2, \ldots, l_d)$ where $l_k$ is $k$-subset of $\{1, \ldots, d\}$

Define

$$a^\mathcal{I} := (a_1 - a_2)1_{l_1} + (a_2 - a_3)1_{l_1} + \cdots + a_d1_{l_d},$$

where $1_{l_i} \in \{0, 1\}^d$ are characteristic vectors.
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where \( \mathbf{1}_{l_j} \in \{0, 1\}^d \) are characteristic vectors.

If \( \mathcal{I} \) is a chain, that is, \( l_1 \subset l_2 \subset \cdots \subset l_d \), then \( a^\mathcal{I} = \sigma \cdot a \) for some \( \sigma \in \mathbb{S}_d \).

Proposition

\[
P_{a,b} = \{ x \in \mathbb{R}^d : \langle a^\mathcal{I}, x \rangle \leq b \text{ for } \mathcal{I} \text{ numerical chain} \}.
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Spectral polyhedra

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Recall that for \( A, B \in S_2\mathbb{R}^d \), \( A \otimes B \) is symmetric \( d^2 \times d^2 \)-matrix with eigenvalues \( \lambda_i(A) \cdot \lambda_j(B) \) for \( 1 \leq i, j \leq d \).
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Theorem

If \( P = P_{a_1,b_1} \cap \cdots \cap P_{a_M,b_M} \) is a symmetric polyhedron, then

\[
\Lambda(P) = \{ A \in S_2 \mathbb{R}^d : b_i \cdot \text{Id} - \hat{\mathcal{L}}_{a_i}(A) \succeq 0 \text{ for } i = 1, \ldots, M \}.
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Spectrahedral shadows

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is a spectrahedron of order \( M \cdot 2^{d^2} \).
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Spectrahedral shadow is a projection of spectrahedron

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In contrast to polyhedra, spectrahedral shadows are in general not spectrahedra.
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Using a result of Ben-Tal and Nemirovski, we can show

**Theorem**

If \( K \subset \mathbb{R}^d \) is a symmetric spectrahedral shadow of order \( r \), then \( \Lambda(P) \) is the projection of a spectrahedron of order \( r + 2d^2 - 2d - 2 \).

Similarly, if \( K \) is a symmetric polyhedron with \( r \) orbits of facets.
Spectral zonotopes and spectral arrangements?

Line segment $[-z, z]$ for $z \in \mathbb{R}^d \setminus 0$
Spectral zonotopes and spectral arrangements?

Line segment $[-z, z]$ for $z \in \mathbb{R}^d \setminus 0$

Zonotope

$$Z = [-z_1, z_1] + \cdots + [-z_m, z_m]$$
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Important in convex geometry, geometric combinatorics, spline theory, ...

The standard permutahedron for \(p = (1, 2, \ldots, d)\)

\[ \Pi(p) = \text{conv}(S_d \cdot p) = p + \sum_{1 \leq i < j \leq d} [-(e_i - e_j), e_i - e_j] \]

Nice formulas for volumes and Steiner polynomials, ...
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Important in convex geometry, geometric combinatorics, spline theory, ...

The standard permutahedron for \(p = (1, 2, \ldots, d)\)

\[ \Pi(p) = \text{conv}(S_d \cdot p) = p + \sum_{1 \leq i < j \leq d} [-\langle e_i - e_j, e_i - e_j \rangle, \langle e_i - e_j, e_i - e_j \rangle] \]

Nice formulas for volumes and Steiner polynomials, ...

Encode arrangement of hyperplanes \(H_i = z_i^\perp\) for \(i = 1, \ldots, m\)

If \(Z\) is a symmetric zonotope, then \(\Lambda(Z)\) is a spectral zonotope.

Encode spectral arrangements

\[ \{ A \in S_2 \mathbb{R}^d : \langle \sigma \cdot z_i, \lambda(A) \rangle = 0 \text{ for some } \sigma \in S_d \} \]

Question

Is there a nice theory of spectral zonotopes and spectral arrangements?
Spectral convex bodies

A spectral convex set is a set of the form

$$\Lambda(K) := \{ A \in S_2 \mathbb{R}^d : \lambda(A) \in K \},$$

where $K$ is a symmetric convex set.

**Proposition**

$\Lambda(K)$ is a convex set/body.

- Rich class of convex sets: Closed under intersection, Minkowski sum, convex hull, and polarity.
- Geometric and algebraic structure: Intimately related to that of $K$; Support functions, orbits of faces, algebraic boundary, hyperbolicity.
- Representations as (projections of) spectrahedra: Spectrahedra when $K$ is a polyhedron; spectrahedral shadow when $K$ is spectrahedral shadow.